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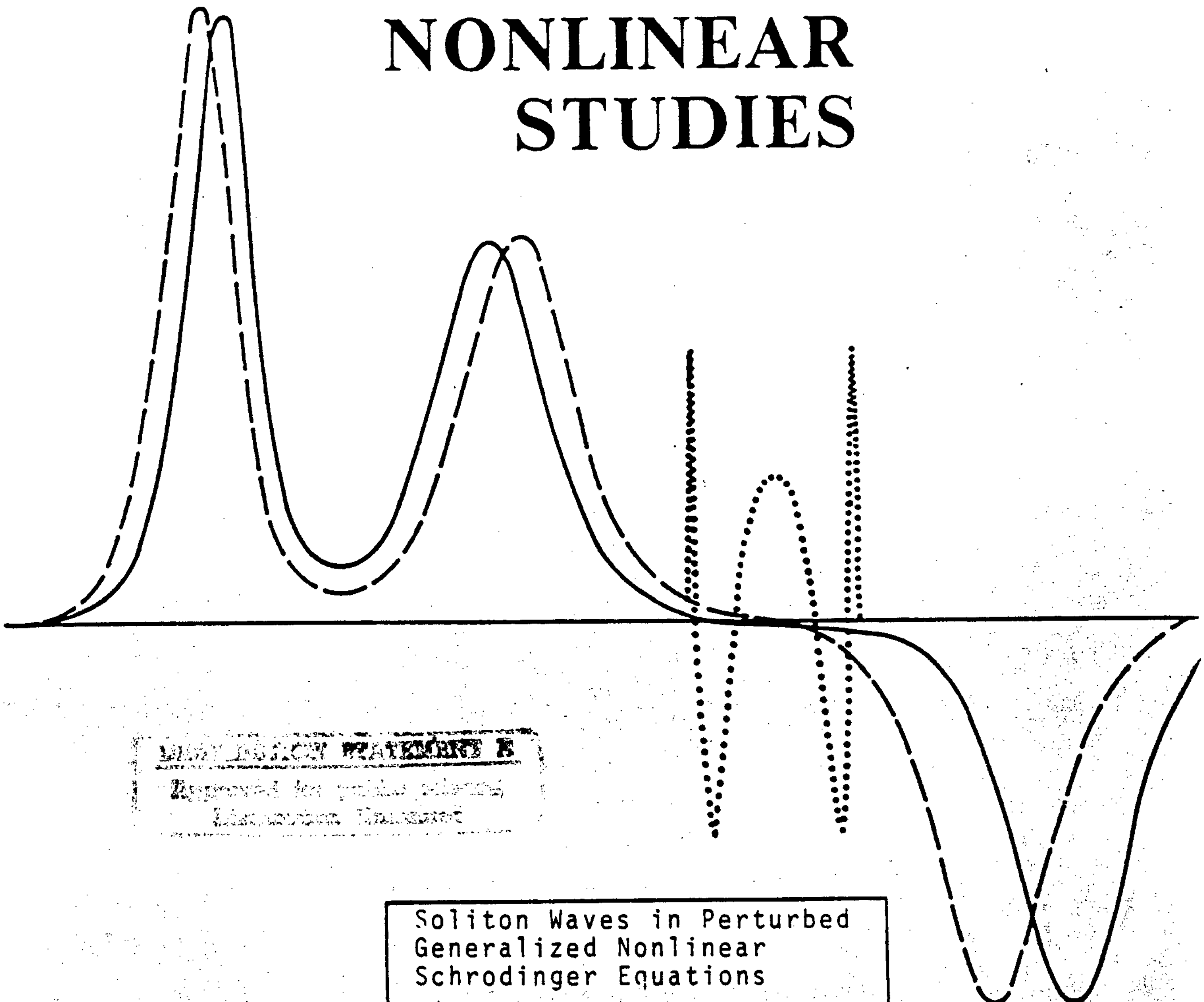

Soliton Waves in Perturbed Generalized Nonlinear Schrodinger Equations

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Abstract

In this paper, we study the stability and evolution of the solitary waves in perturbed generalized nonlinear Schrödinger equations. Our method is based on the completeness of the bounded eigenstates of the associated linear operator in L_2 space and a standard multiple-scale perturbation technique. Unlike the adiabatic perturbation method, ours uncovers all the instability mechanisms in the perturbed equations. As an example, we consider the perturbed cubic-quintic nonlinear Schrödinger equation in detail and determine the stability regions of its solitary waves. The generalization of this method to other perturbed nonlinear wave systems is also discussed.

Keywords: The perturbed generalized nonlinear Schrödinger equations; solitary waves; stability.

1 Introduction

In recent years, the perturbed generalized nonlinear Schrödinger equation has attracted a great deal of attention. This equation is of the form

$$iA_t + A_{xx} + f(|A|^2)A = \epsilon p(A, A^*), \tag{1.1}$$

where f is a real-valued algebraic function, p is a spatial differential operator, and ϵ is a small parameter. It has been shown to govern the evolution of a wave packet in a weakly nonlinear and dispersive medium and has thus arisen in diverse fields such as water waves, plasma and nonlinear

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optics [1, 2, 3, 4, 5]. In particular, this equation is now widely accepted in optics field as a good model for optical pulse propagation in nonlinear fibers (see [6, 7] and the references therein). The rapid advances in optical-soliton based fast-rate telecommunication systems in recent years has stimulated intensive research on it. Another application of Eq. (1.1) is in pattern formation, where it has been used to model some nonequilibrium pattern forming systems (see [8] and the references therein). The unperturbed form of this equation ($\epsilon = 0$) supports solitary waves of the form

$$A = e^{iVx/2+i(\omega-V^2/4)t-i\rho_0}a_0(x-Vt-x_0), \tag{1.2}$$

where $a_0(\theta)$ is a real-valued function and satisfies the equation

$$a_{0\theta\theta} - \omega a_0 + f(a_0^2)a_0 = 0, \tag{1.3a}$$

$$a_0 \longrightarrow 0, \quad |\theta| \to \infty, \tag{1.3b}$$

and $V, \omega(>0), x_0$ and ρ_0 are arbitrary real constants. Note that the existence of these solitary waves is the basis for telecommunication systems using optical solitons as information bits. When perturbations are present, one very important concern is whether these solitary waves will persist or not. This question has been studied extensively in the literature. The linear stability of the solitary waves in the unperturbed equation (1.1) has been investigated in [9, 10], where a criteria for instability was given. When $f(x) = x$, Eq. (1.1) is the perturbed nonlinear Schrödinger equation, whose solitary waves were examined in numerous articles such as [7, 11, 12, 13, 14, 15] among others. The dynamical behavior of the solitary waves in this equation is now well understood. But for more general forms of f, the results are few and far from complete. This case has been investigated in [8, 16, 17, 18] by various methods. In [8], the adiabatic perturbation technique was employed (see also [13]). These authors assumed a quasi-stationary form for the solitary wave, determined the slow evolution of the parameters of this wave and then discovered certain types of instability from those evolution equations. But as they pointed out, the stability they established only refers to the particular class of perturbations compatible with the quasi-stationary solution assumption. It was recognized that there could be other instability mechanisms which could not be found by this adiabatic method (this is indeed the case). In [16], the Evans function approach was used. By calculating the small eigenvalues bifurcating from the zero eigenvalue of the associated linear operator in the unperturbed equation (1.1), the author gave the conditions for this type of instability. His results are basically equivalent to those in [8]. We would like to emphasize here that both results in [8] and [16] missed certain types of instability hidden in Eq. (1.1). The numerical approach to this problem was taken in [17, 18]. In this work, the authors investigated the stability of analytic solitary waves of the cubic-quintic complex Ginzburg-Landau equation and found that they are generally unstable, except in a few special cases. The instability was caused by the existence of growing disturbances whose largest growth rates were numerically estimated. The authors also obtained parameter regions in which stable solitary waves exist for various choices of parameter values. In their work, ϵ was not small, and it was not clear just what was the source of their instability.

In this paper, we develop a new analytical method for studying the stability and evolution of the solitary waves in Eq. (1.1). This method is based on knowing the closure of the bounded eigenstates of the associated linear operator in L_2 space, combined with a standard multiple-scale perturbation method. L_2 is the space of all the square-integrable functions. In essence, this method is similar to the one developed in [12] (see also [14, 15]) for solitons in perturbed integrable equations. But here the new feature is that, since the unperturbed equation (1.1) is non-integrable in general, the completeness of the bounded eigenstates (or equivalently, the Green's function) of the associated linear operator has to be established anew. We will use a direct scattering technique analogous to that in [20, 21] to accomplish this task. In this process, the structure of the spectra of this linear operator will also be obtained and detailed. Using this new method, we can uncover all instabilities of the solitary waves in Eq. (1.1) and give a complete account of the stability and evolution of the solitary waves of this equation. We would like to point out here that, in principle, this method can be applied to any perturbed nonlinear wave equation for uncovering all the instabilities of its permanent waves. We will come back to this point in section 4.

After the general procedure of this method is introduced, we will apply it to the perturbed cubic-quintic nonlinear Schrödinger equation and carry out the analysis in detail. Assuming the perturbation to only contain terms of the Ginzburg-Landau type (as in [8, 16, 17, 18]), we will show that the perturbed cubic-quintic nonlinear Schrödinger equation allows at most two solitary waves, of which, at most one is stable. We also find that the solitary waves of the model equation (1.1) have three instability mechanisms which are related to perturbations of respectively the zero, non-zero (discrete) and continuous eigenvalues of the associated linear operator in the unperturbed equation (1.1). Of these instabilities, the instability related to perturbations of the non-zero discrete eigenvalues has never been studied before. Its capture requires expansion of the perturbation series of the solution out to second order, ϵ^2. We further derive the necessary and sufficient stability conditions for these solitary waves and specify the regions of parameter space, inside of which, this equation has stable solitary waves. Finally, the generalization of this method to the study of the stability of permanent waves in other nonlinear wave systems is discussed.

2 The Procedure

In this section, we detail the procedure for studying the stability and evolution of the solitary waves (1.2) in Eq. (1.1). For simplicity, we consider the case where the perturbation term, p, is of the form

$$p(A, A^*) = \sum_{k=0}^{n} p_k(|A|^2)\frac{\partial^k A}{\partial x^k}, \tag{2.1}$$

where p_k $(k = 1, \ldots, n)$ are complex functions. This will exclude parametrically forced perturbations (see [22]). But even in such cases, the analysis given here can be readily modified. Anticipating the slow evolution of the free parameters of the solitary wave when a perturbation is present, we write the solution of this equation in the form

$$A = e^{iV\theta/2+i\rho}a(\theta, t, T_1, T_2, \ldots, ;\epsilon), \tag{2.2}$$

where

$$\theta = x - \int_0^t V dt - \theta_0, \quad \rho = \int_0^t (\omega + \frac{V^2}{4})dt - \rho_0, \tag{2.3}$$

and ω, V, θ_0 and ρ_0 are all functions of slow time $T_1 = \epsilon t$, $T_2 = \epsilon^2 t, \ldots$. When Eq. (2.2) is substituted into Eq. (1.1), the equation for a is found to be

$$
\begin{aligned}
ia_t - \omega a + a_{\theta\theta} + f(|a|^2)a = \; & \epsilon F - \epsilon\{ia_{T_1} - i\theta_{0T_1}a_\theta + (\frac{V\theta_{0T_1}}{2} - \frac{V_{T_1}\theta}{2} + \rho_{0T_1})a\} \\
& -\epsilon^2\{ia_{T_2} - i\theta_{0T_2}a_\theta + (\frac{V\theta_{0T_2}}{2} - \frac{V_{T_2}\theta}{2} + \rho_{0T_2})a\} + O(\epsilon^3),
\end{aligned} \tag{2.4}
$$

where

$$F = p(A, A^*)e^{-iV\theta/2-i\rho}. \tag{2.5}$$

To solve this equation, we expand a into a perturbation series

$$a = a_0(\theta) + \epsilon a_1 + \epsilon^2 a_2 + \ldots. \tag{2.6}$$

and take a_0 to satisfy Eq. (1.3). Thus, the zeroth order of (2.4) is now trivially satisfied. At order ϵ, a_1 is governed by the linear equation

$$ia_{1t} - \omega a_1 + a_{1\theta\theta} + r(\theta)a_1 + q(\theta)a_1^* = w_1, \tag{2.7a}$$

$$a_1|_{t=0} = 0. \tag{2.7b}$$

where

$$r = f(a_0^2) + a_0^2 f'(a_0^2), \quad q = a_0^2 f'(a_0^2), \tag{2.8}$$

$$w_1 = F_0 - ia_{0T_1} + i\theta_{0T_1}a_{0\theta} - (\frac{V\theta_{0T_1}}{2} - \frac{V_{T_1}\theta}{2} + \rho_{0T_1})a_0, \tag{2.9}$$

$$F_0 = p(A_0, A_0^*)e^{-iV\theta/2-i\rho}, \tag{2.10}$$

and $A_0 = e^{iV\theta/2+i\rho}a_0$. Note that F_0 appears to have a fast time t dependence, but it actually does not, due to the form of p in Eq. (2.1). This fact will be used in the later analysis. Denoting $U_1 = (a_1, a_1^*)^T$, Eq. (2.7) can be rewritten as

$$(i\partial_t + L)U_1 = (w_1, -w_1^*)^T, \tag{2.11a}$$

$$U_1|_{t=0} = 0, \tag{2.11b}$$

where the linear operator L is

$$L = \sigma_3 \begin{pmatrix} \partial_{\theta\theta} - \omega + r & q \\ q & \partial_{\theta\theta} - \omega + r \end{pmatrix}, \tag{2.12}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.13}$$

are Pauli spin matrices.

The linear operator L is the key to solving this problem. If we could expand all quantities in the eigenstates of this operator, then we can expand and solve (2.4) to all orders. So, first we need the eigenstates, $\psi(x, \lambda)$, where λ is the eigenvalue, and the eigenvalue spectra of the operator L, where

$$L\psi = \lambda\psi. \tag{2.14}$$

We will now discuss the main features of this eigenvalue problem. Due to the form of L, if λ is an eigenvalue and ψ the corresponding eigenfunction, then it follows that $-\lambda$, λ^* and $-\lambda^*$ are also eigenvalues, with corresponding eigenfunctions as $\sigma_1\psi$, ψ^* and $\sigma_1\psi^*$. In the appendix, we detail the structure of the spectra of the operator L, using the direct scattering technique. We show that the discrete eigenvalues of L are zeros of the analytical function Δ_2 defined there. Now, there are four free parameters in the unperturbed solitary wave (1.2). Perturbations of, or shifts in these four free parameters correspond to the four degrees of freedom represented by the degenerate $\lambda = 0$ eigenvalue. Thus $\lambda = 0$ is at least a four-fold eigenvalue of L. Two of these degenerate eigenvalues have two discrete eigenfunctions

$$\psi_{01} = a_{0\theta}(1,1)^T, \quad \psi_{02} = a_0(1,-1)^T, \tag{2.15}$$

which exactly satisfy the eigenvalue equations

$$L\psi_{01} = 0, \quad L\psi_{02} = 0. \tag{2.16}$$

The other two eigenvalues correspond to two generalized eigenfunctions, called "derivative states" in [14, 15] and elsewhere, and are given by

$$\phi_{01} = \frac{1}{2}\theta a_0(1,-1)^T, \quad \phi_{02} = a_{0\omega}(1,1)^T. \tag{2.17}$$

These states are not true eigenfunctions, but they are necessary for closure [23]. They satisfy the modified eigenvalue equations

$$L\phi_{01} = \psi_{01}, \quad L\phi_{02} = \psi_{02}. \tag{2.18}$$

The total number of the discrete eigenvalues of L (with the multiplicity of all degenerate eigenvalues included) is given by the "winding number" of Δ_2, as its argument moves along the path, P, described in the appendix, and shown in Fig. 3. The continuous eigenvalues of L are found along the two half lines $\{\lambda : \lambda > \omega\}$ and $\{\lambda : \lambda < -\omega\}$. For each value of λ in those intervals, there are two continuous eigenstates, one symmetric in θ (denoted as $\psi_s(\theta,\lambda)$) and the other one anti-symmetric in θ (denoted as $\psi_a(\theta,\lambda)$). In the appendix, we also show that the union of the discrete regular eigenstates, the discrete generalized eigenstates and the continuous eigenstates, form a complete set in the space of L_2. Therefore, Eq. (2.11) can be solved by expanding the solution, U_1, and the inhomogeneous term, in this complete set of functions.

For simplicity, in the rest of the paper, we assume that zero is a four-fold discrete eigenvalue of L. In addition, we assume that L has only two other simple, discrete, non-zero eigenvalues, denoted as λ_d and $-\lambda_d$, with the corresponding eigenfunctions denoted by ψ_d and $\psi_{-d} = \sigma_1\psi_d$. This is the case for the perturbed cubic-quintic nonlinear Schrödinger equation, to be discussed in more detail later in this paper. If L has more than the above eigenvalues, or if the non-zero discrete eigenvalues are not simple, then the following analysis can be easily appropriately modified.

Under the above assumptions, we then can expand $(w_1, -w_1^*)^T$ and U_1 in this closed set. Using arbitrary coefficients, we can take

$$\begin{aligned}(w_1, -w_1^*)^T = \; & c_{01}\psi_{01}(\theta) + c_{02}\psi_{02}(\theta) + d_{01}\phi_{01}(\theta) + d_{02}\phi_{02}(\theta) \\ & + c_d\psi_d(\theta) + c_{-d}\psi_{-d}(\theta) + \textstyle\int_I \{c_a(\lambda)\psi_a(\theta,\lambda) + c_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda,\end{aligned} \tag{2.19}$$

$$U_1 = h_{01}\psi_{01}(\theta) + h_{02}\psi_{02}(\theta) + g_{01}\phi_{01}(\theta) + g_{02}\phi_{02}(\theta) + h_d\psi_d(\theta) + h_{-d}\psi_{-d}(\theta) + \int_I \{h_a(\lambda)\psi_a(\theta,\lambda) + h_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda, \tag{2.20}$$

where the interval $I = (-\infty, -\omega] \cup [\omega, \infty)$. We will define an inner product by

$$< f, g >= \int_{-\infty}^{\infty} f^T(\theta)\sigma_3 g(\theta)d\theta. \tag{2.21}$$

Then it is easy to show that the only non-zero inner products of these bounded eigenstates are

$$< \psi_{01}, \phi_{01} >, \ < \psi_{02}, \phi_{02} >, \ < \psi_d, \psi_d >, \ < \psi_{-d}, \psi_{-d} >, < \psi_a, \psi_a >, \text{ and } < \psi_s, \psi_s > .$$

In particular,

$$< \psi_a(\cdot,\lambda), \psi_a(\cdot,\lambda') >= k_a(\lambda)\delta(\lambda - \lambda'), \tag{2.22a}$$

$$< \psi_s(\cdot,\lambda), \psi_s(\cdot,\lambda') >= k_s(\lambda)\delta(\lambda - \lambda'), \tag{2.22b}$$

where $k_a(\lambda)$ and $k_s(\lambda)$ can be related to the scattering data of Eq. (2.14). When the expansions (2.19) and (2.20) are substituted into Eq. (2.11), and the above inner products are used, then we obtain the following equations for the coefficients in U_1:

$$i\frac{\partial h_{01}}{\partial t} + g_{01} = c_{01}, \quad i\frac{\partial h_{02}}{\partial t} + g_{02} = c_{02}, \tag{2.23a}$$

$$i\frac{\partial g_{01}}{\partial t} = d_{01}, \quad i\frac{\partial g_{02}}{\partial t} = d_{02}, \tag{2.23b}$$

$$i\frac{\partial h_d}{\partial t} + \lambda_d h_d = c_d, \quad i\frac{\partial h_{-d}}{\partial t} - \lambda_d h_{-d} = c_{-d}, \tag{2.23c}$$

$$i\frac{\partial h_a}{\partial t} + \lambda h_a = c_a, \quad i\frac{\partial h_s}{\partial t} + \lambda h_s = c_s, \tag{2.23d}$$

$$h_{01} = h_{02} = g_{01} = g_{02} = h_d = h_{-d} = h_a = h_s = 0, \quad \text{at } t = 0. \tag{2.23e}$$

Here

$$c_{01} = \frac{< (w_1, -w_1^*)^T, \phi_{01} >}{< \psi_{01}, \phi_{01} >}, \quad c_{02} = \frac{< (w_1, -w_1^*)^T, \phi_{02} >}{< \psi_{02}, \phi_{02} >}, \tag{2.24a}$$

$$d_{01} = \frac{< (w_1, -w_1^*)^T, \psi_{01} >}{< \psi_{01}, \phi_{01} >}, \quad d_{02} = \frac{< (w_1, -w_1^*)^T, \psi_{02} >}{< \psi_{02}, \phi_{02} >}, \tag{2.24b}$$

$$c_d = \frac{< (w_1, -w_1^*)^T, \psi_d >}{< \psi_d, \psi_d >}, \quad c_{-d} = \frac{< (w_1, -w_1^*)^T, \psi_{-d} >}{< \psi_{-d}, \psi_{-d} >}, \tag{2.24c}$$

$$c_a = \frac{< (w_1, -w_1^*)^T, \psi_a >}{k_a}, \quad c_s = \frac{< (w_1, -w_1^*)^T, \psi_s >}{k_s}. \tag{2.24d}$$

Note that $c_{-d} = -c_d^*$ since $\psi_{-d} = \sigma_1\psi_d$ and is real. For the same reason, $h_{-d} = h_d^*$. Since w_1 does not depend on the fast time t, the quantities in Eqs. (2.24) do not either. To suppress the secular terms in h_{01}, h_{02}, g_{01} and g_{02}, we need to require that

$$c_{01} = c_{02} = d_{01} = d_{02} = 0. \tag{2.25}$$

In view of Eqs. (2.24) and (2.9), these four conditions will produce the following slow evolution equations for V, ω, θ_0 and ρ_0 on T_1 time scale:

$$\frac{dV}{dT_1} = \frac{4\int_{-\infty}^{\infty} a_{0\theta}\ \mathrm{Re}(F_0)d\theta}{\int_{-\infty}^{\infty} a_0^2\ d\theta}, \tag{2.26a}$$

$$\frac{d\omega}{dT_1} = \frac{2\int_{-\infty}^{\infty} a_0\ \mathrm{Im}(F_0)d\theta}{\int_{-\infty}^{\infty}(a_0^2)_\omega d\theta}, \tag{2.26b}$$

$$\frac{d\theta_0}{dT_1}\int_{-\infty}^{\infty} a_0^2\ d\theta + \frac{d\omega}{dT_1}\int_{-\infty}^{\infty}\theta(a_0^2)_\omega d\theta = 2\int_{-\infty}^{\infty}\theta a_0\ \mathrm{Im}(F_0)d\theta, \tag{2.26c}$$

$$(V\frac{d\theta_0}{dT_1} + 2\frac{d\rho_0}{dT_1})\int_{-\infty}^{\infty}(a_0^2)_\omega d\theta - \frac{dV}{dT_1}\int_{-\infty}^{\infty}\theta(a_0^2)_\omega d\theta = 4\int_{-\infty}^{\infty} a_{0\omega}\mathrm{Re}(F_0)d\theta. \tag{2.26d}$$

Here "Re" and "Im" represent the real and imaginary parts of a complex number. It is noted that Eqs. (2.26a, b) have been obtained before by the adiabatic perturbation method in [8]. Similar equations were also derived for solitons in perturbed nonlinear Schrödinger equations (see [7, 11, 15]). In order for the solitary waves of the model (1.1) to be stable, these equations must have stable fixed points. Otherwise, an instability will arise. Such an instability would be due to the zero eigenvalue, of the linear operator L, bifurcating and moving into the unstable region, because of the perturbations. This has been discussed in [8, 16].

When conditions (2.26) are satisfied, solving Eq. (2.23), we get

$$h_{01} = h_{02} = g_{01} = g_{02} = 0, \tag{2.27a}$$

$$h_d = c_d\{1 - \alpha_d(T_1)e^{i\lambda_d t}\}/\lambda_d, \quad h_{-d} = h_d^*, \tag{2.27b}$$

$$h_a = c_a\{1 - \alpha_a(T_1)e^{i\lambda t}\}/\lambda, \tag{2.27c}$$

$$h_s = c_s\{1 - \alpha_s(T_1)e^{i\lambda t}\}/\lambda, \tag{2.27d}$$

and

$$\alpha_d(0) = \alpha_a(0) = \alpha_s(0) = 1. \tag{2.28}$$

Then the solution U_1 is

$$U_1 = h_d\psi_d(\theta) + h_{-d}\psi_{-d}(\theta) + \int_I\{h_a(\lambda)\psi_a(\theta,\lambda) + h_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda. \tag{2.29}$$

Here the α's are constants of the integration, and possibly functions of T_1, as indicated. The c's are slowly varying with T_1 when V and ω are.

It is important to realize here that, in order for the solitary wave (1.2) to be stable, in addition to the conditions (2.25), we also need to require that the coefficients h_d, h_{-d}, h_a and h_s in U_1 do not grow unbounded, on either the t or the T_1 scales. On t scale, h_a and h_s are already bounded since the continuous eigenvalues of the operator L are always real (see Eq. (2.27 c, d)). But in order for h_d and h_{-d} to remain bounded on this scale, it is necessary for λ_d to be real. If this is not so, then the solitary wave is unstable. We now shall assume that λ_d is real.

On the T_1 scale, we need to ensure that α_d, α_a and α_s in Eq. (2.27) remain bounded. To obtain the evolution equations for these coefficients, we need to expand (2.4) out to second order, ϵ^2. When Eq. (2.6) is substituted into Eq. (2.4) and terms of order ϵ^2 collected, an equation for a_2 will be obtained. Denoting $U_2 = (a_2, a_2^*)^T$, the equation for U_2 is

$$(i\partial_t + L)U_2 = (w_2, -w_2^*)^T, \tag{2.30a}$$

$$U_2|_{t=0} = 0, \tag{2.30b}$$

where

$$\begin{aligned} w_2 = \; & F_1 - ia_{1T_1} + i\theta_{0T_1}a_{1\theta} - (V\theta_{0T_1}/2 - V_{T_1}\theta/2 + \rho_{0T_1})a_1 \\ & -ia_{0T_2} + i\theta_{0T_2}a_{0\theta} - (V\theta_{0T_2}/2 - V_{T_2}\theta/2 + \rho_{0T_2})a_0 \\ & -a_0 f'(a_0^2)a_1(a_1 + 2a_1^*) - a_0^3 f''(a_0^2)(a_1 + a_1^*)^2/2, \end{aligned} \tag{2.31}$$

$$F_1 = \{p_A(A_0, A_0^*)A_1 + p_{A^*}(A_0, A_0^*)A_1^*\}e^{-iV\theta/2-i\rho}, \tag{2.32}$$

and $A_1 = e^{iV\theta/2+i\rho}a_1$. This equation can be solved analogous to Eq. (2.11). We expand $(w_2, -w_2^*)^T$ and U_2 as

$$\begin{aligned} (w_2, -w_2^*)^T = \; & \hat{c}_{01}\psi_{01}(\theta) + \hat{c}_{02}\psi_{02}(\theta) + \hat{d}_{01}\phi_{01}(\theta) + \hat{d}_{02}\phi_{02}(\theta) \\ & +\hat{c}_d\psi_d(\theta) + \hat{c}_{-d}\psi_{-d}(\theta) + \textstyle\int_I\{\hat{c}_a(\lambda)\psi_a(\theta,\lambda) + \hat{c}_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda, \end{aligned} \tag{2.33}$$

and

$$\begin{aligned} U_2 = \; & \hat{h}_{01}\psi_{01}(\theta) + \hat{h}_{02}\psi_{02}(\theta) + \hat{g}_{01}\phi_{01}(\theta) + \hat{g}_{02}\phi_{02}(\theta) \\ & +\hat{h}_d\psi_d(\theta) + \hat{h}_{-d}\psi_{-d}(\theta) + \textstyle\int_I\{\hat{h}_a(\lambda)\psi_a(\theta,\lambda) + \hat{h}_s(\lambda)\psi_s(\theta,\lambda)\}d\lambda. \end{aligned} \tag{2.34}$$

The coefficients in U_2 are governed by equations similar to (2.23) with only a hat added to each quantity. To suppress the secular terms in $\hat{h}_{01}$, $\hat{h}_{02}$, $\hat{g}_{01}$ and $\hat{g}_{02}$, we will obtain the evolution equations for the parameters V, ω, θ_0 and ρ_0 on T_2 timescale. These equations are negligible in deference to Eqs. (2.26) which govern their evolutions on T_1 timescale. The coefficient $\hat{h}_d$ in U_2 is governed by the equation

$$i\frac{\partial \hat{h}_d}{\partial t} + \lambda_d \hat{h}_d = \hat{c}_d, \tag{2.35}$$

where

$$\hat{c}_d = \frac{< (w_2, -w_2^*)^T, \psi_d >}{< \psi_d, \psi_d >}. \tag{2.36}$$

Now $\hat{c}_d$ has resonant terms which are proportional to $e^{i\lambda_d t}$. To see this, we put Eq. (2.1) into (2.32) and get

$$F_1 = \sum_{k=0}^{n}\{p_k'(a_0^2)\frac{\partial^k A_0}{\partial\theta^k}(A_0^*A_1 + A_0A_1^*) + p_k(a_0^2)\frac{\partial^k A_1}{\partial\theta^k}\}e^{-iV\theta/2-i\rho}. \tag{2.37}$$

When Eqs. (2.27) and (2.29) are substituted into Eq. (2.37), we find that the $e^{i\lambda_d t}$ and $e^{-i\lambda_d t}$ coefficients in F_1 are proportional to $c_d\alpha_d/\lambda_d$ and $(c_d\alpha_d)^*/\lambda_d$ respectively. Suppose such terms in F_1 are

$$\frac{c_d\alpha_d}{\lambda_d}e_1(\theta)e^{i\lambda_d t} + [\frac{c_d\alpha_d}{\lambda_d}e_2(\theta)]^*e^{-i\lambda_d t}, \tag{2.38}$$

then the coefficient of the $e^{i\lambda_d t}$ term in $\hat{c}_d$ is found, from Eq. (2.36), to be

$$K = \{i(c_d\alpha_d)_{T_1} + (k_1 + k_2 + k_3)c_d\alpha_d\}/\lambda_d, \tag{2.39}$$

where

$$k_1 = (V\theta_{0T_1}/2 - V_{T_1}\theta/2 + \rho_{0T_1})\frac{\int_{-\infty}^{\infty}(\psi_{d1}^2 + \psi_{d2}^2)d\theta}{\int_{-\infty}^{\infty}(\psi_{d1}^2 - \psi_{d2}^2)d\theta}, \tag{2.40a}$$

$$k_2 = \frac{(c_d + c_d^*)}{\lambda_d}\frac{\int_{-\infty}^{\infty}(\psi_{d1} + \psi_{d2})\{2a_0 f'(a_0^2)(\psi_{d1}^2 + \psi_{d1}\psi_{d2} + \psi_{d2}^2) + a_0^3 f''(a_0^2)(\psi_{d1} + \psi_{d2})^2\}d\theta}{\int_{-\infty}^{\infty}(\psi_{d1}^2 - \psi_{d2}^2)d\theta}, \tag{2.40b}$$

$$k_3 = \frac{\int_{-\infty}^{\infty}(e_1\psi_{d1} + e_2\psi_{d2})d\theta}{\int_{-\infty}^{\infty}(\psi_{d1}^2 - \psi_{d2}^2)d\theta}, \tag{2.40c}$$

and e_1 and e_2 were introduced in (2.38). Since $Ke^{i\lambda_d t}$ term in $\hat{c}_d$ is a homogeneous solution of Eq. (2.35), then in order to suppress the secular growth in $\hat{h}_d$, we must have $K = 0$. This gives us a slow evolution equation for $c_d\alpha_d$, which is

$$\frac{d(c_d\alpha_d)}{dT_1} = i(k_1 + k_2 + k_3)c_d\alpha_d. \tag{2.41}$$

Note that both k_1 and k_2 are real quantities. Thus if $\mathrm{Im}(k_3)$ is negative, $c_d\alpha_d$ will exponentially grow, and the solitary wave (1.2) will be unstable. This instability is caused by the pumping of the solitary wave energy into the discrete eigenmodes $\psi_{\pm d}$ when the perturbation is turned on. It can also be interpreted as the initially real discrete eigenvalues, $\pm\lambda_d$, moving into an unstable region in the presence of perturbations. In fact, λ_d is shifted to $\lambda_d + \epsilon(k_1 + k_2 + k_3)$ in view of Eq. (2.27b). This instability was never analyzed before in the literature. Similar argument applies to the coefficients $\hat{h}_a$ and $\hat{h}_s$ of the continuous eigenstates in U_2. Suppression of the secular terms in those coefficients will produce evolution equations for $\alpha_a(T_1, \lambda)$ and $\alpha_s(T_1, \lambda)$ on T_1 scale. If α_a or α_s grows unbounded, instability will also arise. This instability is caused by energy being injected into the continuous eigenmodes of L, or the continuous eigenvalues of L moving into the unstable region, under perturbations. In summary, by studying the evolution equations of the solitary wave parameters and the coefficients in the U_1 solution, all the instability mechanisms can be, and have been now uncovered.

In the above, we have obtained the slow evolution equations for $V, \omega, \theta_0, \rho_0$ and α_d. The equations for α_a and α_s in the coefficients of the continuous eigenmodes of U_1 are more troublesome. The reason is that these equations involve convolutions which couple together α_a and α_s over all the continuous eigenvalues. To circumvent this difficulty, it is helpful to view this type of instability as due to the continuous eigenvalues of L moving into the unstable region. Note that the continuous eigenvalues of the linearization operator around a solitary wave, even for the perturbed equation (1.1) as well as for the unperturbed version, can easily be specified (see [24]). Thus this type of instability can be determined without the necessity of deriving and examining the evolution equations for α_a and α_s. With this hurdle removed, then our procedure, as described above, can be carried out, and the stability regions of the solitary waves under perturbations can be specified.

3 The Perturbed Cubic-Quintic Nonlinear Schödinger Equation

In this section, we use the perturbed cubic-quintic nonlinear Schödinger equation of Ginzburg-Landau type as an example and carry out the detailed analysis. This equation is of the form

$$iA_t + A_{xx} + c_3|A|^2A + c_5|A|^4A = \epsilon i(b_1 A_{xx} + \gamma A - b_3|A|^2A - b_5|A|^4A), \tag{3.1}$$

where all the coefficients are real-valued, and $c_3 = \pm 1$ by scaling. When $\epsilon = 0$, Eq. (3.1) supports solitary waves of the form (1.2), where

$$a_0(\theta) = [\frac{4\omega}{c_3 + \sqrt{1 + 16c_5\omega/3}\ \cosh(2\sqrt{\omega}\ \theta)}]^{1/2}. \tag{3.2}$$

If $c_3 = 1$, this wave exists when $c_5\omega > -3/16$; if $c_3 = -1$, it exists when $c_5 > 0$. The linear operator L is given by Eqs. (2.8) and (2.12) where $f(x) = c_3x + c_5x^2$.

We first establish the spectrum structure of L. We have shown that its continuous eigenvalues are the intervals $I = (-\infty, -\omega] \cup [\omega, \infty)$. To determine the total number of its discrete eigenvalues, we numerically calculated $\triangle_2$ along the path P shown in the appendix for $c_3 = \pm 1$ and c_5, ω being allowed various values. For each of the three cases: (1) $c_3 = 1, c_5 > 0$, (2) $c_3 = 1, c_5 < 0$, and (3) $c_3 = -1$ ($c_5 > 0$), the results are always qualitatively the same. In case (2), the orbit of $\triangle_2$, as ζ moves along P, is sketched in Fig. 1. We see that in this case, the winding number of $\triangle_2$ is four, thus L has four discrete eigenvalues (multiplicity of eigenvalues included). In the other two cases, we find that the winding number of $\triangle_2$ is six. Recall that $\lambda = 0$ is always a discrete eigenvalue of L. To determine its multiplicity, we chose a small closed path around $\lambda = 0$ (i.e. $\zeta = e^{i\pi/4}$) and find that the winding number of $\triangle_2$ is always four, for all three cases. This means that $\lambda = 0$ is always a four-fold discrete eigenvalue of L. We then conclude that in case (2), $\lambda = 0$ is the only discrete eigenvalue of L, while in the other two cases, L has two additional non-zero, discrete eigenvalues. Due to the symmetry of the eigenvalues, these two non-zero eigenvalues have to be either real or purely imaginary. In addition, one is always the negative of the other. We will denote them as λ_d and $-\lambda_d$ as before. Closer examination reveals that in case (1), λ_d is real, and in case (3), it is purely imaginary. This is consistent with the results in [9, 10]. It indicates that in case (1), the solitary wave (1.2) is linearly neutrally stable in the unperturbed equation (3.1). But in case (3), it is linearly unstable, thus also unstable under weak perturbations. In the rest of the paper, we assume $c_3 = 1$.

Next we determine the slow evolution equations for the solitary wave parameters; V, ω, θ_0 and ρ_0. In view of the perturbation term in Eq. (3.1), and after some algebra, we find that Eqs. (2.26) become

$$\frac{dV}{dT_1} = -\frac{4b_1\int_{-\infty}^{\infty} a_{0\theta}^2 d\theta}{\int_{-\infty}^{\infty} a_0^2 d\theta} V, \tag{3.3}$$

$$\frac{d\omega}{dT_1} = \frac{S(\omega) - b_1V^2/2}{\frac{d}{d\omega}\ln\int_{-\infty}^{\infty} a_0^2 d\theta}, \tag{3.4}$$

$$\frac{d\theta_0}{dT_1} = \frac{d\rho_0}{dT_1} = 0, \tag{3.5}$$

where

$$S(\omega) = (2\gamma - b_1\omega - \frac{3b_5\omega}{c_5}) + (\frac{b_1}{4} - 2b_3 + \frac{9b_5}{4c_5})\frac{\int_{-\infty}^{\infty} a_0^4 d\theta}{\int_{-\infty}^{\infty} a_0^2 d\theta}. \tag{3.6}$$

The fixed points of Eqs. (3.3) and (3.4) are $V = 0$ and

$$S(\omega) = 0. \tag{3.7}$$

In order for the fixed points to be stable, we need to require that

$$b_1 > 0, \quad \frac{S'(\omega)}{\frac{d}{d\omega}\ln\int_{-\infty}^{\infty} a_0^2 d\theta} < 0. \tag{3.8}$$

Explicit expressions for $S(\omega)$ and $\int_{-\infty}^{\infty} a_0^2 d\theta$ can be obtained depending on the sign of c_5.

1. $c_5 > 0$: In this case,

$$\int_{-\infty}^{\infty} a_0^2 d\theta = \sqrt{3/c_5}\,(\pi/2 - \arctan u^{-1}), \tag{3.9}$$

$$S(\omega) = -\frac{3}{16c_5^2}(s_1 + s_2\omega + \frac{s_3 u}{\pi/2 - \arctan u^{-1}}), \tag{3.10}$$

where

$$s_1 = b_1c_5 - 8b_3c_5 + 9b_5 - 32\gamma c_5^2/3, \tag{3.11a}$$

$$s_2 = 16c_5(b_5 + b_1c_5/3), \tag{3.11b}$$

$$s_3 = -b_1c_5 + 8b_3c_5 - 9b_5, \tag{3.11c}$$

and

$$u = \sqrt{16c_5\omega/3}. \tag{3.12}$$

It is easy to check that $S''(\omega)$ does not change sign for $\omega > 0$, hence the concavity of $S(\omega)$ does not change. As a result, Eq. (3.7) has at most two fixed points and Eq. (3.1) allows at most two solitary waves. When Eq. (3.7) has two fixed points, $S'(\omega)$ will have opposite signs at these points. From Eq. (3.9) we see that the sign of $\frac{d}{d\omega}\ln\int_{-\infty}^{\infty} a_0^2 d\theta$ is always positive, thus one of these two fixed points is stable, and the other unstable. We can also readily show that, if $b_5 + b_1c_5/3 < 0$, Eq. (3.4) has one unstable fixed point; if $b_5 + b_1c_5/3 > 0$, it has none or two fixed points (one stable and the other one unstable).

2. $c_5 < 0$: In this case, similar results can be obtained. Here

$$\int_{-\infty}^{\infty} a_0^2 d\theta = \sqrt{-3/c_5}\,\tanh^{-1} v, \tag{3.13}$$

$$S(\omega) = -\frac{3}{16c_5^2}(s_1 + s_2\omega + \frac{s_3 v}{\tanh^{-1} v}), \tag{3.14}$$

where s_i ($i = 1, 2, 3$) are given in Eq. (3.11), and $v = \sqrt{-16c_5\omega/3}$. Similarly, we can show that Eq. (3.1) has at most two solitary waves, at most one stable. Furthermore, if $3b_5 - 4b_3c_5 - 16\gamma c_5^2/3 < 0$, Eq. (3.4) has one unstable fixed point; if $3b_5 - 4b_3c_5 - 16\gamma c_5^2/3 > 0$, it has none or two fixed points (one stable and the other unstable).

When Eqs. (3.3) and (3.4) allow stable fixed points, the corresponding solitary wave may still be unstable due to the non-zero discrete eigenvalues moving into the unstable region under perturbations. We have shown that when $c_5 < 0$, non-zero discrete eigenvalues do not exist, but when $c_5 > 0$, two such eigenvalues of opposite sign exist and are real. Suppose λ_d and $-\lambda_d$ are these two non-zero eigenvalues, and $\psi_d = (\psi_{d1}, \psi_{d2})^T$ and $\psi_{-d} = \sigma_1 \psi_d$ the corresponding eigenfunctions. Upon inserting the perturbation terms of Eq. (3.1) into Eq. (2.40c) and after some simplifications, we find that

$$\mathrm{Im}(k_3) = b_1(m_1 + V^2/4) - \gamma + 2m_3 b_3 + 3m_5 b_5, \tag{3.15}$$

where

$$m_1 = \frac{\int_{-\infty}^{\infty} (\psi_{d1\theta}^2 - \psi_{d2\theta}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{3.16a}$$

$$m_3 = \frac{\int_{-\infty}^{\infty} a_0^2 (\psi_{d1}^2 - \psi_{d2}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}, \tag{3.16b}$$

and

$$m_5 = \frac{\int_{-\infty}^{\infty} a_0^4 (\psi_{d1}^2 - \psi_{d2}^2) d\theta}{\int_{-\infty}^{\infty} (\psi_{d1}^2 - \psi_{d2}^2) d\theta}. \tag{3.16c}$$

If we take $V(=0)$ and ω as the stable fixed points of their evolution equations (3.3) and (3.4), then $\mathrm{Im}(k_3)$ in Eq. (3.15) can be evaluated. If it is negative, according to Eq. (2.41), the solitary wave in Eq. (3.1) will be unstable.

Lastly, we consider the instability of the solitary waves (1.2) in Eq. (3.1) caused by the continuous eigenvalues of L under perturbations. In this case, the results in [24] indicate that when $b_1 < 0$ or $\gamma > 0$, these continuous eigenvalues will move into the unstable region due to perturbations. Otherwise, this type of instability is absent.

Now we summarize the above results on the stability of the solitary waves (1.2) in the perturbed cubic-quintic nonlinear Schrödinger equation (3.1). When $c_3 = -1$, all the solitary waves are unstable. When $c_3 = 1$ and $c_5 < 0$, the solitary wave is stable if and only if $b_1 > 0$, $\gamma < 0$, $V = 0$ and ω is the stable fixed point of Eq. (3.4). When $c_3 = 1$ and $c_5 > 0$, it is stable if and only if $b_1 > 0$, $\gamma < 0$, $V = 0$, ω is the stable fixed point of Eq. (3.4), and $\mathrm{Im}(k_3)$ given by Eq. (3.15) is positive. Comparison of these results with those by the adiabatic perturbation method [8] shows that, when $c_3 = 1$ and $c_5 < 0$, the adiabatic method yields the correct stability conditions; but when $c_3 = 1$ and $c_5 > 0$, it does not. The reason is that it misses the instability caused by the additional mode with the non-zero discrete eigenvalue, λ_d, into which the solitary wave could emit energy, as manifested by Eq. (2.41).

As two examples, we choose $c_3 = 1, c_5 = 1$ or -1, $\gamma = -0.1$, $b_3 = -1$ and determine the regions in the (b_1, b_5) plane, for the existence of stable solitary waves by using the above results. For the given b_1 and b_5 values, we first numerically determine the stable fixed point ω from Eqs. (3.7) and (3.8) using Newton's method. This alone will give the region of stable solitary waves for $c_5 = -1$. It is shown in Fig. 2 (II). When $c_5 = 1$, we take the stable fixed point, ω, of Eq. (3.7), and numerically determine the discrete eigenmode λ_d and ψ_d from Eq. (2.14), using the shooting method. We then evaluate m_k $(k = 1, 2, 3)$ from Eqs. (3.16) and $\mathrm{Im}(k_3)$ from Eq. (3.15), with V being taken as zero. The stable region is the set of (b_1, b_5) points where the stable fixed point, ω, exists and

$\mathrm{Im}(k_3)$ is positive. This region is shown in Fig. 2 (I). At any point in these regions, Eq. (3.1) has exactly one stable solitary wave. We observe that in both cases, when the diffusive term (b_1) in the perturbation increases, for the solitary wave to be stable, the nonlinear absorption (b_5) has to decrease, but not be below a certain lower bound. An interesting fact is that, in the $c_5 = 1$ case, stable solitary waves exist even when $b_5 < 0$ (see Fig. 2 (I)). In this case, both the nonlinear effects in the perturbations are amplifying, but they are offset by strong diffusion. Thus a stable pulse is still possible. However, such stable regions are very small. Note that in Fig. 2 (I), inside the region enclosed by the two solid curves and the b_5 axis, Eqs. (3.3) and (3.4) have a unique stable fixed point. This is the region captured by the adiabatic perturbation method [8]. But the solitary wave is still unstable in most of this region due to instability of eigenvalue λ_d bifurcations.

4 Discussion

In this paper, we studied the stability and evolution of the solitary waves in perturbed generalized nonlinear Schrödinger equation (1.1), and the perturbed cubic-quintic nonlinear Schrödinger equation of Ginsburg-Landau type (3.1) in particular. We found that the solitary waves in Eq. (1.1) are subject to three types of instability which are associated with the bifurcations of the zero, non-zero (discrete), and continuous eigenvalues of the linear operator L in the presence of perturbations. Our stability conditions for the solitary waves are both necessary and sufficient. When specializing to the perturbed cubic-quintic nonlinear Schrödinger equation of Ginzburg-Landau type, we proved that for any set of parameters, Eq. (3.1) has at most one stable solitary wave. We also specified the parameter regions of stable solitary waves and graphed them for two particular examples. When compared to the results in [8] by the adiabatic perturbation method, we found that the adiabatic method missed certain types of instability, especially the one associated with the bifurcation of the non-zero discrete eigenvalues of the operator L.

The method we employed in this work is based on the completeness of the bounded eigenstates of the operator L and a standard multiple-scale perturbation technique. The key in this analysis is the completeness of L's bounded eigenstates in L_2 space. It allowed us to solve the relevant linearized equations at various orders and detect secularities in the linear solutions, which then set the stage for the multiple-scale perturbation method to come into play. For general perturbed nonlinear wave systems, if this completeness of the bounded eigenstates of the associated linear operator can be established, then the analysis in this paper can be adapted to those systems as well and a full account of the stability and evolution of permanent waves in the presence of perturbations can be provided. The completeness of the bounded eigenstates of a linear operator has been studied extensively in the literature (see [20, 25, 26] for example). It has been well established for self-adjoint operators. For generic non-self-adjoint operators, as discussed in the appendix, we can prove the completeness using the direct scattering technique similar to that in [20]. For general operators, corresponding to discrete eigenvalues, generalized eigenstates as well as the regular eigenstates may exist. But if the set of the discrete eigenvalues is finite, we can still show that the eigenstates and the generalized eigenstates of the linear operator form a complete set. The details will be discussed elsewhere. In this light, our recipe for the study of stability and evolution of permanent waves in perturbed nonlinear systems can be widely applied.

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Appendix

In this appendix, we study the spectrum structure of the operator L given by Eq. (2.12), and establish the completeness of its bounded eigenstates in L_2 space. For the exactly integrable nonlinear Schrödinger equation, the eigenstates of L are related to the squared Zakharov-Shabat eigenstates. Thus the completeness of L's bounded eigenfunctions can be established by the inverse scattering technique [15, 27]. But for the generalized nonlinear Schrödinger equation, that connection breaks down. In this case, we will use the direct scattering method as developed in [20, 21] to accomplish this task. For convenience, we will replace θ by x. We first consider the general potentials $q(x)$ and $r(x)$ which vanish at infinity, then specialize to the present case where q and r are given by Eq. (2.8).

The eigenvalue problem

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix} \tag{A1}$$

can be written out as

$$u_{xx} - (\omega+\lambda)u = -r(x)u - q(x)v, \tag{A2a}$$

$$v_{xx} - (\omega-\lambda)v = -r(x)v - q(x)u. \tag{A2b}$$

To avoid dealing with the branch cuts at $\lambda = \pm\omega$, we make the following parameter transform

$$\lambda = \omega(\zeta^2 + \zeta^{-2})/2. \tag{A3}$$

Then Eq. (A2) becomes

$$Y_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \delta^2 - r & 0 & -q & 0 \\ 0 & 0 & 0 & 1 \\ -q & 0 & \eta^2 - r & 0 \end{pmatrix} Y, \tag{A4}$$

where $Y = (u, u_x, v, v_x)^T$, and

$$\delta = \sqrt{\omega/2}\,(\zeta + \zeta^{-1}), \quad \eta = i\sqrt{\omega/2}\,(\zeta - \zeta^{-1}). \tag{A5}$$

The rest of the analysis is analogous to that for a n-th order scalar scattering problem considered in [20]. Thus the results will only be sketched here with the proofs omitted. We define the singular set Σ as

$$\Sigma = \{\zeta : \text{the real parts of any two numbers of } \delta, -\delta, \eta \text{ and } -\eta \text{ are equal}\}. \tag{A6}$$

It is easy to see that Σ is the set of all the rays originating from $\zeta = 0$ with angles being the multiples of $\pi/4$. We number these rays cyclically as $\Sigma_0, \Sigma_1, \ldots$ and the sectors $C \setminus \Sigma$ as $\Omega_1, \Omega_2, \ldots$ (shown in Fig. 3). On Σ_0, Re(η)=0; on Σ_1, Re(δ) =Re($-\eta$) ($\neq 0$); on Σ_2, Re(δ)=0; in sector Ω_1, Re(δ)>Re($-\eta$)>Re(η)>Re($-\delta$); in Ω_2, Re($-\eta$)> Re(δ)>Re($-\delta$)>Re(η); etc. In each sector, we define two fundamental matrices Φ^- and Φ^- of Eq. (A4) according to the ordering of $\delta, -\delta, \eta$ and $-\eta$ in that sector. For instance, in Ω_1, we define

$$\Phi^{\mp}(x, \zeta) = m^{\mp}(x, \zeta)e^{xJ}, \tag{A7}$$

where $J = \mathrm{diag}(\delta, -\eta, \eta, -\delta)$,

$$m^{\mp}(x, \zeta) \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ \delta & 0 & 0 & -\delta \\ 0 & 1 & 1 & 0 \\ 0 & -\eta & \eta & 0 \end{pmatrix}, \quad x \to \pm\infty, \tag{A8}$$

and $m^{\mp}$ are bounded as $|x| \to \infty$. In other sectors, $\Phi^{\pm}$ can be similarly defined. It is easy to see that $\Phi^{\mp}$ so defined are unique. They exist for all $\zeta \in C \setminus \Sigma$, apart from a discrete set Z which are all the zeros of $\triangle_k$ ($k = 1, 2, 3$) to be defined below. At the points $\zeta_k \in Z$, $\Phi^{\pm}$ have pole singularity.

Next we define the functions

$$\triangle_1 = m_1^- \wedge m_2^+ \wedge m_3^+ \wedge m_4^+/(-4\delta\eta), \tag{A9a}$$

$$\triangle_2 = m_1^- \wedge m_2^- \wedge m_3^+ \wedge m_4^+/(-4\delta\eta), \tag{A9b}$$

$$\triangle_3 = m_1^- \wedge m_2^- \wedge m_3^- \wedge m_4^+/(-4\delta\eta), \tag{A9c}$$

where "$\wedge$" represents the wedge products of vectors. It can be shown that $\triangle_k$ ($k = 1, 2, 3$) are independent of x, analytic in each sector Ω_i, and $\triangle_k \to 1$ as $|\zeta| \to \infty$. Furthermore, $\triangle_2$ is analytic across the boundary Σ_1. The discrete eigenvalues λ of the operator L correspond to the zeros of $\triangle_2$ through relation (A3). The continuous eigenvalues of L correspond to the two rays Σ_0 and Σ_2. We choose a path P in the ζ plane as shown in Fig. 3, with its direction counter-clockwise. This path starts at $\zeta = 0^+ + \infty i$, moves down vertically to $\zeta = i$, half-circles around it, and moves downward again until it reaches $\zeta = 0$. Then it quarter-circles around $\zeta = 0$ and moves horizontally along the upper side of the positive Re(ζ) axis, until it arrives at $\zeta = 1$. Then it half-circles around $\zeta = 1$, keeps on moving horizontally, and eventually ends at $\zeta = \infty + 0^- i$. In the λ plane, this path corresponds to one which encloses the entire λ plane except the continuous spectrum $\{\lambda : \lambda > \omega \text{ or } \lambda < -\omega\}$. Thus the winding number of $\triangle_2$ along P:

$$N = \frac{1}{2\pi i}\int_P \frac{\triangle_2'(\zeta)}{\triangle_2(\zeta)}d\zeta = \frac{1}{2\pi}\{\arg\{\triangle_2(\infty) - \arg\{\triangle_2(\infty + i)\}\} \tag{A10}$$

gives the total number of the discrete eigenvalues of L (multiplicity of non-simple eigenvalues counted).

The completeness of the bounded eigenstates of L in L_2 space can be established by constructing the Green's function to the equation

$$(L - \lambda)G(x, y, \zeta) = \delta_y(x)\mathrm{diag}(1, -1), \tag{A11}$$

and proving that G can be expressed as a linear combination of L's bounded eigenstates. We call the operator L generic if (1) $\triangle_k$ ($k = 1, 2, 3$) have no common zeros and no multiple zeros; (2) they have no zeros on Σ, and (3) the set of their zeros is finite. For a generic self-adjoint operator, the completeness of its bounded eigenstates in L_2 space was proved in [20] using this approach. In our case, L is not self-adjoint. But if it is generic, slight modification to the analysis in [20] can be made to establish the completeness relation as well. If L is non-generic (as is the case when $q(x)$ and $r(x)$ are given by Eq. (2.8)), a discrete eigenvalue may be multi-fold, and its algebraic multiplicity may be larger than its geometric multiplicity. In such cases, the generalized eigenstates of the discrete eigenvalues are also needed for closure (see [15]). But as long as the number of L's discrete eigenvalues is finite, similar techniques can be used to show that the bounded eigenstates of L (including the generalized discrete eigenstates) also form a complete set. As a practical guide, if a discrete eigenvalue is a k-th fold root of $\triangle_2$, then correspondingly k regular or generalized eigenstates should be included.

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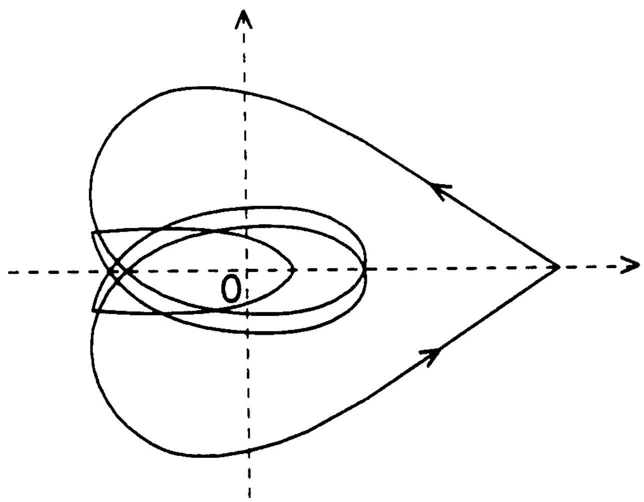

Figure 1: The trajectory of $\triangle_2(\zeta)$ as ζ moves along the path P for $c_3 = 1$ and $c_5 < 0$. The path P is specified in the appendix and shown in Fig. 3.

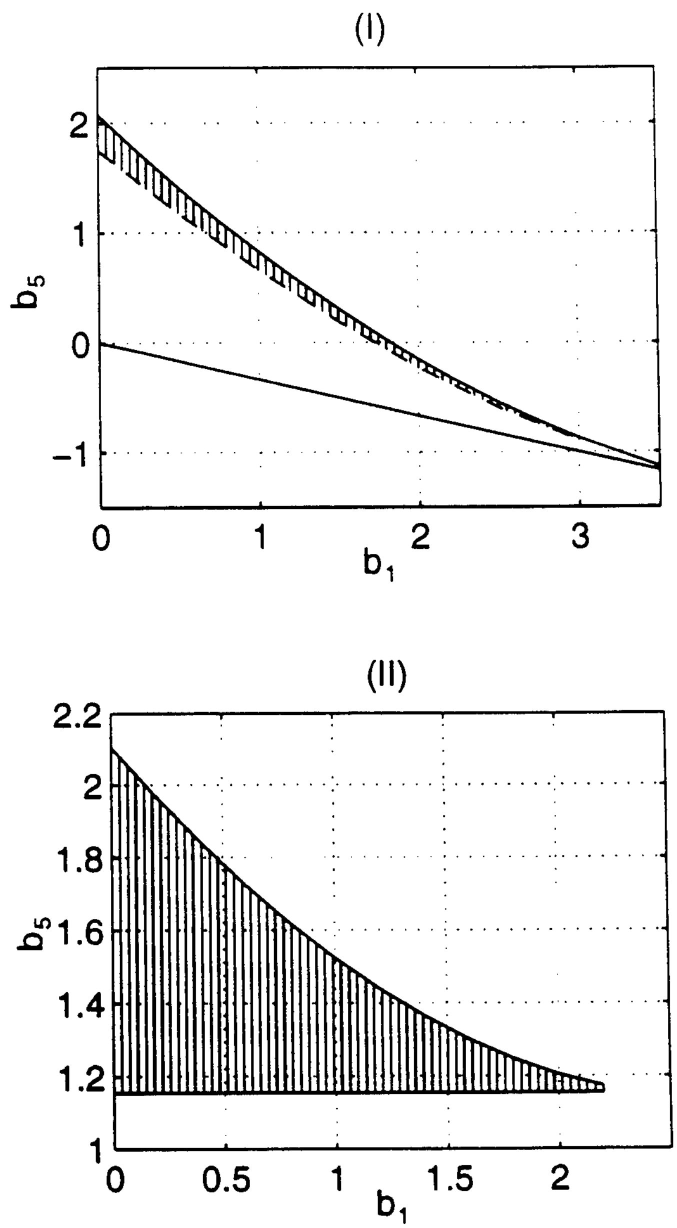

Figure 2: Stability regions (shaded) of the solitary waves in Eq. (3.1) for $c_3 = 1, \gamma = -0.1$ and $b_3 = -1$. (I) $c_5 = 1$; (II) $c_5 = -1$.

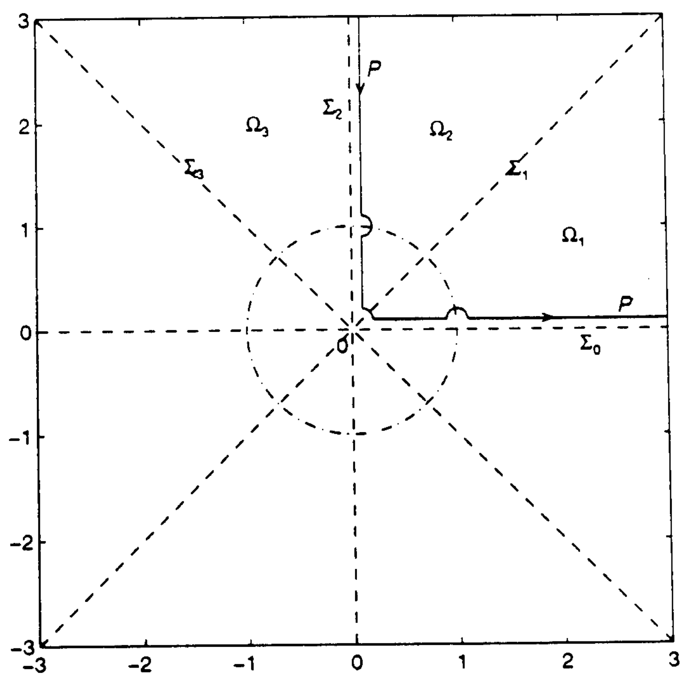

Figure 3: The complex ζ-plane.